Particle differential equation explanation

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02/01/2020

A problem I had when creating the particle explosion was getting the particles to precisely reach the edge of the screen. I thought about it for a while, before realizing that the particles could be described by a simple differential equation. The particles start with a set speed, and that speed gets multiplied with a constant $0 < f < 1$, where f is the friction acting on the particle. The particle therefore has an acceleration equal to $-v(1 - f)$, where v is the particle's speed and f is the friction. As both acceleration and velocity are a result of deriving position, we can create a differential equation: (p is the particle's position)

$$a = -v(1 - f)$$
$$p'' = -p'(1 - f)$$

We also know that the particle has a start position of 0, and a start velocity of v. Using all of this we can find the solution to the differential equation:

$$p(t) = \frac{v - v \cdot e^{-ft}}{f}$$

This function describes the particle's position at a given time t. What we're interested in is the total distance traveled, the limit of the position as t approaches ∞. Logically, this limit only exists if the friction is greater than 0 and less than 1, so it's somewhat hard to make a computer compute the limit. Luckily it's easy to find the solution manually:

$$\begin{aligned} d &= \lim_{t\to\infty} \frac{v - v \cdot e^{-ft}}{f} \\ &= \frac{v - v \cdot \lim_{t\to\infty}(e^{-tf})}{f} \end{aligned}$$

$$\lim_{t\to\infty}(e^{-tf}) = e^{-\infty} = 0 \qquad \text{—} \quad 0 < f < 1$$

$$\begin{aligned} d &= \lim_{t\to\infty} \frac{v - v \cdot e^{-ft}}{f} \\ &= \frac{v - v \cdot 0}{f} \\ &= \frac{v}{f} \end{aligned}$$

This gives an equation for the distance traveled given velocity and friction, the only step remaining is solving the equation for the velocity, and then we're done:

$$d = \frac{v}{f}$$
$$\implies v = d \cdot f$$

This simple and elegant equation is what you can find in the code. Given how simple it is, I assume there is a much simpler way of finding it, but I think my way was quite fun.